

# Note on the saturation of the norm inequalities between diamond and nuclear norm

Ulrich Michel, Martin Kliesch, Richard Kueng, and David Gross

## Abstract

The diamond norm plays an important role in quantum information and operator theory. Recently, it has also been proposed as a regularizer for low-rank matrix recovery. The norm constants that bound the diamond norm in terms of the nuclear norm (also known as trace norm) are explicitly known. This note provides a simple characterization of all operators saturating the upper and the lower bound.

## I. INTRODUCTION

The *diamond norm* [1], [2] plays an important role in quantum information and in operator theory, where it equals – essentially<sup>1</sup> – the *norm of complete boundedness* or *cb-norm*. Recently, some of the present authors proposed its use as a regularizer for low-rank matrix recovery in the context of compressed sensing [5]. In particular, it was proven that for operators saturating a certain norm inequality, diamond norm regularization performs at least as well as (and heuristically better than) the usual nuclear norm regularization [5]. The set of operators saturating this norm inequality was explicitly characterized [5].

This previous work based its analysis on a semidefinite programming (SDP) formulation [3] for the diamond norm—henceforth called *Watrous’ SDP*. The SDP-formulation renders the proof quite technical.

In this note we present a simple and compact proof of this characterization. Moreover, we characterize also all matrices saturating a converse norm inequality.

### A. Notation

The space of linear maps on  $\mathcal{V}$  is denoted by  $L(\mathcal{V})$  and the identity by  $\mathbb{1}_{\mathcal{V}} \in L(\mathcal{V})$ . A nonnegative real number  $\sigma$  is called a *singular value* of an operator  $A \in L(\mathcal{V})$  if and only if, there exist normalized vectors  $u, v \in \mathcal{V}$  such that  $Av = \sigma u$  and  $A^\dagger u = \sigma v$ . We call such vectors  $u, v$  *left singular* and *right singular vector* for  $\sigma$ , respectively. The *operator square root* of a positive semidefinite operator  $A \succeq 0$  is defined to be the unique positive semidefinite operator  $\sqrt{A}$  obeying  $\sqrt{A}^2 = A$ . We will use the nuclear norm  $\|\cdot\|_1$ , Frobenius norm  $\|\cdot\|_2$  and spectral norm  $\|\cdot\|_\infty$  defined by

$$\|A\|_1 := \text{Tr} \sqrt{A^\dagger A}, \quad \|A\|_2 := \sqrt{\text{Tr}[A^\dagger A]}, \quad \|A\|_\infty := \sup_{x \in \mathcal{V}} \frac{\|Ax\|_2}{\|x\|_2}. \quad (1)$$

Following [5], we define the *square norm* of a *bipartite operator*  $X \in L(\mathcal{W} \otimes \mathcal{V})$  to be

$$\|X\|_\square = \sup_{A, B \in L(\mathcal{V})} \left\{ \|(\mathbb{1}_{\mathcal{W}} \otimes A) X (\mathbb{1}_{\mathcal{W}} \otimes B)\|_1, \|A\|_2 = \|B\|_2 = \sqrt{\dim(\mathcal{V})} \right\}. \quad (2)$$

The square norm is closely related [5] to the diamond norm  $\|\cdot\|_\diamond$  by  $\|J(M)\|_\square = \dim(\mathcal{V}) \|M\|_\diamond$ , where  $M \in L(\mathcal{V}) \rightarrow L(\mathcal{W})$  is a linear map,  $J$  is the Choi-Jamiołkowski isomorphism [6], [7] (so that  $J(M) \in L(\mathcal{W} \otimes \mathcal{V})$ ). Finally, the partial trace  $\text{Tr}_{\mathcal{W}} : L(\mathcal{W} \otimes \mathcal{V}) \rightarrow L(\mathcal{V})$  is the linear map defined on product operators  $(A \otimes B) \in L(\mathcal{W} \otimes \mathcal{V})$  by  $\text{Tr}_{\mathcal{W}}[A \otimes B] = \text{Tr}[A]B$  and linearly extended to all  $L(\mathcal{W} \otimes \mathcal{V})$ .

All authors are with the Institute for Theoretical Physics, University of Cologne, Germany. M. Kliesch is also with the Institute of Theoretical Physics and Astrophysics, University of Gdańsk, Poland. D. Gross is also with the Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, Australia.

<sup>1</sup> The diamond norm of a map is the cb-norm of its adjoint  $\|X\|_\diamond = \|X^\dagger\|_{\text{cb}}$  [3], [4].

## II. RESULT

The following equivalence of norms holds (see, e.g., [5]):

$$\|X\|_1 \leq \|X\|_\square \leq \dim(\mathcal{V}) \|X\|_1. \quad (3)$$

We provide a characterization of (i) all operators saturating the upper bound and (ii) the lower bound on  $\|X\|_\square$ .

**Theorem 1** (Saturation of the lower bound [5]). *Let  $X \in L(\mathcal{W} \otimes \mathcal{V})$ . Then  $\|X\|_1 = \|X\|_\square$  if and only if*

$$\text{Tr}_{\mathcal{W}}[\sqrt{XX^\dagger}] = \text{Tr}_{\mathcal{W}}[\sqrt{X^\dagger X}] = \frac{\|X\|_1}{\dim(\mathcal{V})} \mathbb{1}_{\mathcal{V}}. \quad (4)$$

This theorem was proven in Ref. [5] using Watrous' SDP [3] and complementary slackness. Below we provide a shorter and more direct proof. The similar upper bound is saturated as follows.

**Theorem 2** (Saturation of the upper bound). *Let  $X \in L(\mathcal{W} \otimes \mathcal{V})$ . Then  $\|X\|_\square = \dim(\mathcal{V}) \|X\|_1$  if and only if there exist rank-1 projectors  $\phi\phi^\dagger, \psi\psi^\dagger \in L(\mathcal{V})$  such that*

$$\text{Tr}_{\mathcal{W}}[\sqrt{XX^\dagger}] = \|X\|_1 \psi\psi^\dagger \quad \text{and} \quad (5)$$

$$\text{Tr}_{\mathcal{W}}[\sqrt{X^\dagger X}] = \|X\|_1 \phi\phi^\dagger. \quad (6)$$

## III. PROOFS

In this section we first state our auxiliary statements and then the proofs of our theorems.

### A. Auxiliary statements

The concept of *sign functions* of real numbers is extendable to non-Hermitian matrices.

**Definition 3** (Sign matrix). *For any matrix  $X \in L(\mathcal{V})$  with singular value decomposition  $X = U\Sigma V^\dagger$  we define its sign matrix to be  $S_X := VU^\dagger$ .*

Note that the sign matrix obeys

$$XS_X = U\Sigma U^\dagger = \sqrt{XX^\dagger}, \quad (7)$$

$$S_X X = V\Sigma V^\dagger = \sqrt{X^\dagger X}. \quad (8)$$

The eigenvectors of  $XS_X$  and  $S_X X$  are *left singular* and *right singular vectors* of  $X$ , respectively. The saturation results from our theorems are derived using similar saturation results for Schatten norms. For any matrix  $X \in L(\mathbb{C}^n)$  holds that  $\|X\|_1 \leq \sqrt{n} \|X\|_2$  and we observe that this bound is saturated for unitary operators.

**Observation 4** (Saturation: 1 and 2-norm). *Let  $X \in L(\mathcal{V})$ .*

*Then  $\|X\|_2 = \frac{1}{\sqrt{n}} \|X\|_1$  if and only if  $\frac{\sqrt{n}}{\|X\|_1} X$  is unitary.*

Schatten norms satisfy Hölder's inequality, i.e., for  $A, B \in L(\mathcal{V})$  it holds that  $\|AB\|_1 \leq \|A\|_\infty \|B\|_1$ . The saturation condition involves the range  $\text{ran}(B)$  of  $B$ .

**Proposition 5** (Saturation of Hölder's inequality). *Let  $A, B \in L(\mathcal{V})$ . Then the following are equivalent:*

- i) *Tightness in Hölder's inequality, i.e.  $\|AB\|_1 = \|A\|_\infty \|B\|_1$*
- ii) *Every vector  $x \in \text{ran}(B)$  is a right singular vector of  $A$  with singular value  $\|A\|_\infty$*
- iii) *The restriction of  $\frac{A}{\|A\|_\infty}$  to  $\text{ran}(B)$  is an isometry.*

A proof of this proposition is provided in the Appendix. Applying it twice yields a similar result for the iterated version of Hölder's inequality  $\|ABC\|_1 \leq \|A\|_\infty \|B\|_1 \|C\|_\infty$  (for  $A, B, C \in L(\mathbb{C}^n)$ ).

**Observation 6** (Saturation of iterated Hölder's inequality). *Let be  $A, B, C \in L(\mathbb{C}^n)$ .*

*Then  $\|ABC\|_1 = \|A\|_\infty \|B\|_1 \|C\|_\infty$  if and only if*

- i) *Every vector  $x \in \text{ran}(B)$  is a right singular vector of  $A$  with singular value  $\|A\|_\infty$  and*
- ii) *Every vector  $y \in \text{ran}(B^\dagger)$  is a left singular vector of  $C$  with singular value  $\|C\|_\infty$ .*

Directly from the definition of the partial trace follows that

$$\text{Tr}[X(\mathbb{1}_{\mathcal{W}} \otimes B)] = \text{Tr}[\text{Tr}_{\mathcal{W}}(X)B] \quad (9)$$

for all  $X \in \text{L}(\mathcal{W} \otimes \mathcal{V})$  and  $B \in \text{L}(\mathcal{V})$ .

### B. Proofs

Now we use the previous observations to prove our theorems.

*Proof of Theorem 1.* We recall the definition of the square norm,

$$\|X\|_{\square} = \sup_{A, B \in \text{L}(\mathcal{V})} \left\{ \|(\mathbb{1}_{\mathcal{W}} \otimes A) X (\mathbb{1}_{\mathcal{W}} \otimes B)\|_1, \|A\|_2 = \|B\|_2 = \sqrt{\dim(\mathcal{V})} \right\}. \quad (10)$$

For convenience we define  $\tilde{X}_A := (\mathbb{1}_{\mathcal{W}} \otimes A) X$  and  $n := \dim(\mathcal{V})$ . Using the unitary invariance of the nuclear norm, the fact that  $\text{Tr}[Y] \leq \|Y\|_1$ , and Eq. (9) we obtain

$$\|(\mathbb{1}_{\mathcal{W}} \otimes A) X (\mathbb{1}_{\mathcal{W}} \otimes B)\|_1 \geq \text{Tr}[S_{\tilde{X}_A} \tilde{X}_A (\mathbb{1}_{\mathcal{W}} \otimes B)] = \text{Tr}\left[\sqrt{\tilde{X}_A^\dagger \tilde{X}_A} (\mathbb{1}_{\mathcal{W}} \otimes B)\right] \quad (11)$$

$$= \text{Tr}\left[\text{Tr}_{\mathcal{W}}\left[\sqrt{\tilde{X}_A^\dagger \tilde{X}_A}\right] B\right]. \quad (12)$$

Choosing  $B \propto \text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A}$  (normalized as  $\|B\|_2 = \sqrt{n}$ ), which will turn out to yield the optimal value, we obtain

$$\|X\|_{\square} \geq \sup_{A \in \text{L}(\mathcal{V})} \left\{ \sqrt{n} \left\| \text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A} \right\|_2, \|A\|_2 = \sqrt{n} \right\}. \quad (13)$$

Using that  $\sqrt{n} \|Y\|_2 \geq \|Y\|_1$  for all  $Y \in \text{L}(\mathcal{V})$  yields

$$\sqrt{n} \left\| \text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A} \right\|_2 \geq \left\| \text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A} \right\|_1 = \|(\mathbb{1}_{\mathcal{W}} \otimes A) X\|_1. \quad (14)$$

Analogously as before, we proceed by using the unitary invariance of the 1-norm and the sign  $S_X$  from the right to obtain

$$\sqrt{n} \left\| \text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A} \right\|_2 \geq \|(\mathbb{1}_{\mathcal{W}} \otimes A) \sqrt{X X^\dagger}\|_1. \quad (15)$$

We choose  $A \propto \text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger}$  and combine it with the bound (13) to finally arrive at

$$\|X\|_{\square} \geq \sqrt{n} \left\| \text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger} \right\|_2 \geq \left\| \text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger} \right\|_1 = \|X\|_1. \quad (16)$$

The equality of both norms holds if and only if all inequalities are saturated. In particular we have that

$$\sqrt{n} \left\| \text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger} \right\|_2 = \left\| \text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger} \right\|_1 \quad (17)$$

in the bound (16) if and only if  $\text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger}$  is unitary up to a normalization factor (Observation 4). Since  $\text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger}$  is also positive semidefinite we obtain  $\text{Tr}_{\mathcal{W}} \sqrt{X X^\dagger} = \frac{\|X\|_1}{n} \mathbb{1}$  as the only possible operator. From the bound (14) follows, with the same argument, that  $\text{Tr}_{\mathcal{W}} \sqrt{\tilde{X}_A^\dagger \tilde{X}_A} \propto \mathbb{1}$ . This relation implies the second desired identity  $\text{Tr}_{\mathcal{W}} \sqrt{X^\dagger X} = \frac{\|X\|_1}{n} \mathbb{1}$ .  $\square$

*Proof of Theorem 2.* " $\Rightarrow$ ": We use the hypothesis and split the 1-norm of products with an iterated Hölder's inequality in order to maximize separately over  $A$  and  $B$  in the definition of the square norm (2),

$$n \|X\|_1 = \|X\|_{\square} \leq \sup_{A, B \in \text{L}(\mathcal{V})} \left\{ \|\mathbb{1}_{\mathcal{W}} \otimes A\|_{\infty} \|X\|_1 \|\mathbb{1}_{\mathcal{W}} \otimes B\|_{\infty}, \|A\|_2 = \|B\|_2 = \sqrt{n} \right\} = n \|X\|_1. \quad (18)$$

The largest possible value is attained with rank 1 matrices  $A = \sqrt{n} \nu \psi^\dagger$ ,  $B = \sqrt{n} \phi \mu^\dagger$  for normalized  $\nu, \mu, \phi, \psi \in \mathcal{V}$ . Since the inequality is saturated we conclude with Observation 6 that  $X = Y \otimes \psi \phi^\dagger$  for an arbitrary non-zero operator  $Y \in \text{L}(\mathcal{W})$ , and the result follows.

" $\Leftarrow$ ": From the hypothesis we have that  $X = Y \otimes \psi \phi^\dagger$ . Instead of maximizing in the definition of the square norm (2) we just choose  $A \propto \psi \psi^\dagger$ ,  $B \propto \phi \phi^\dagger$ . This choice yields

$$n \|X\|_1 \geq \|X\|_{\square} \geq n \|(\mathbb{1}_{\mathcal{W}} \otimes \psi \psi^\dagger) (Y \otimes \psi \phi^\dagger) (\mathbb{1}_{\mathcal{W}} \otimes \phi \phi^\dagger)\|_1 = n \|X\|_1 \quad (19)$$

and the result follows.  $\square$

## ACKNOWLEDGEMENTS

The work of UM was supported by the BCGS under Grant GSC 260. The research of MK is financed by the Polonez grant 2015/19/P/ST2/03001. RK and DG acknowledge financial support from the Excellence Initiative of the German Federal and State Governments (Grant ZUK 81), the ARO under contract W911NF-14-1-0098 (Quantum Characterization, Verification, and Validation), and the DFG (SPP1798 CoSIP).

## REFERENCES

- [1] A. Y. Kitaev, A. Shen, and M. N. Vyalyi, *Classical and quantum computation*. American Mathematical Society, 2002, vol. 47.
- [2] J. Watrous, “CS 766 theory of quantum information,” Available online at <https://cs.uwaterloo.ca/~watrous/LectureNotes.html>, 2011, accessed: 2016-09-13.
- [3] J. Watrous, “Simpler semidefinite programs for completely bounded norms,” Jul. 2012, arXiv:1207.5726.
- [4] V. Paulsen, *Completely bounded maps and operator algebras*. Cambridge University Press, 2002, vol. 78.
- [5] M. Kliesch, R. Kueng, J. Eisert, and D. Gross, “Improving compressed sensing with the diamond norm,” *IEEE Trans. Inf. Th.*, vol. 62, pp. 7445 – 7463, 2016.
- [6] M.-D. Choi, “Completely positive linear maps on complex matrices,” *Lin. Alg. App.*, vol. 10, no. 3, pp. 285 – 290, 1975.
- [7] A. Jamiolkowski, “Linear transformations which preserve trace and positive semidefiniteness of operators,” *Rep. Math. Phys.*, vol. 3, no. 4, pp. 275 – 278, 1972.
- [8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. New York, NY, USA: Cambridge University Press, 1991.

## APPENDIX

In this appendix we provide the proof of the saturation condition for Hölder’s inequality.

*Proof of Proposition 5.*

“ii)  $\Rightarrow$  iii)”:  $\frac{A}{\|A\|_\infty}$  maps an arbitrary  $\phi \in \text{ran}(B)$  from one singular basis to another. Hence, restricted to  $\text{ran}(B)$  it is an isometry.

“iii)  $\Rightarrow$  i)”: We represent the isometry  $\frac{A}{\|A\|_\infty}$  with a unitary matrix  $U$  that fulfills  $\|A\|_\infty UB = AB$ . Using the unitary invariance of the nuclear norm, we obtain  $\|AB\|_1 = \|A\|_\infty \|UB\|_1 = \|A\|_\infty \|B\|_1$ .

“i)  $\Rightarrow$  ii)”: We denote the  $i$ -th largest singular value of a matrix  $X$  with  $\sigma_i(X)$ . We upper bound the nuclear norm by splitting the singular values of the matrix as [8, Problem 3.3.4]

$$\|AB\|_1 = \sum_{i=1}^n \sigma_i(AB) \leq \sum_{i=1}^n \sigma_i(A) \sigma_i(B). \quad (20)$$

Upper bounding the singular values of  $A$  by the largest singular value yields

$$\sum_{i=1}^n \sigma_i(A) \sigma_i(B) \leq \sum_{i=1}^{\text{rank}(B)} \|A\|_\infty \sigma_i(B) = \|A\|_\infty \|B\|_1. \quad (21)$$

By assumption, every inequality is saturated. In particular, the largest  $\text{rank}(B)$  singular values of  $A$  are all equal. Furthermore,  $\sigma_i(AB) = \|A\|_\infty \sigma_i(B)$  and, thus, all vectors  $x \in \text{ran}(B)$  are right singular vectors of  $A$  with singular value  $\|A\|_\infty$ .  $\square$